International Journal of Theoretical Physics, Vol. 37, No. 5, 1998

New Solitary Wave Solution of the Combined KdV and mKdV Equation

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Received August 9, 1997

This paper presents two different methods for the construction of new exact solitary wave solutions to the combined KdV and mKdV equation. There exist 12 types of soliton solutions which reduce to those of the mKdV equation. There exist three types of solurion solutions which reduce to those of the KdV equation.

1. INTRODUCTION

The combined KdV and mKdV equation

$$u_t + 6auu_x + 6bu^1u_x + cu_{xxx} = 0 \tag{1}$$

with constants a, b, and c, is widely used in various fields such as solid—state physics, plasma physics, fluid physics, and quantum field theory (Wadati, 1975a, b; Dey, 1986; Konno and Ichikawa, 1974; Narayanamurti and Varma, 1970; Tappert and Varma, 1970). It is clear that (1) is a combination of the KdV and mKdV equations. As a result the combined KdV and mKdV equation is also integrable, which means that it has a Backlund transformation, a bilinear form, a Lax pair, and an infinite number of conservation laws (Miura, 1968), etc. Exact solutions for (1) have been obtained by Wadati (1975a, b) by means of the inverse scattering transform and by Hirota's method. More recently, Coffey (1990), Mohamad (1992), and Lou and Chen (1994) found the trvaelling wave solutions of (1) by the series expansion method, a direct method, and a mapping approach. In this paper, by adopt two different methods, we give 12 types of new solitary wave solutions of (1).

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0020-7748/98/0500-154 1$15.00/0

2. THE LEADING-ORDER ANALYSIS METHOD

The travelling wave solutions of (1) take the form

$$u(\bar{x}, t) = u(\xi), \qquad \xi = k(x - wt) \tag{2}$$

where k and w are constants to be determined later. After substituting (2) into (1), we find that

$$-wu_\xi + 6auu_\xi + 6bu^2 u_\xi + ck^2 u_{\xi\xi\xi} = 0 \tag{3}$$

Noticing the properties of the functions $f(\xi) = \tanh \xi$, $\coth \xi$, $\mathrm{tg}\, \xi$, $\mathrm{ctg}\, \xi$

$$\frac{d}{d\xi} \sum_{j=0}^{s} A_j f^j(\xi) = \sum_{j=0}^{s+1} B_j f^j(\xi) \tag{4}$$

$$\left(\sum_{j=0}^{s} A_j f^j(\xi)\right)^2 = \sum_{j=0}^{2s} C_j f^j(\xi) \tag{5}$$

where A_j ($j = 0, 1, \ldots, n$), B_j ($j = 0, 1, \ldots, n + 1$), and C_j ($j = 0, 1, \ldots, 2n$) are constants, we make the crucial ansatz

$$u(\xi) = \sum_{j=0}^{s} A_j f^j(\xi) \tag{6}$$

where A_i are constants.

On balacing the highest order contributions from the linear terms with the highest order contribution from the nonlinear terms, we find that $m = 1$, and thus

$$u(\xi) = A_0 + A_1 f(\xi) \tag{7}$$

By substituting equation (7) into (3) and equating the coefficients of functions $f^j(\xi)$ ($j = 0, 1, \ldots, 4$), we obtain the following three independent equations.

(i) For $f(\xi) = \tanh \xi$ or $\coth \xi$,

$$a + 2bA_0 = 0 \tag{8}$$

$$w - 6aA_0 + 6bA_1^2 - 6bA_0^2 + 8ck^2 = 0 \tag{9}$$

$$w - 6aA_0 - 6bA_0^2 + 2ck^2 = 0 \tag{10}$$

We have four unknowns A_0, A, k, w and three equations and so may choose w as a free parameter. From (8)–(10), the other variables are then found to be

$$A_0 = -\frac{a}{2b} \tag{11}$$

$$A_1 = \pm\sqrt{\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)} \tag{12}$$

$$k = \pm\sqrt{\frac{-1}{2c}\left(w + \frac{3a^2}{2b}\right)} \tag{13}$$

Considering tanh $(-\zeta) = -$ tanh ζ and coth $(-\zeta) = -$ coth ζ, the solitary wave solutions of (1) read

$$u_1(x,t) = \sqrt{\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\tanh\left[\frac{1}{2}\sqrt{-\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{14}$$

$$u_2(x,t) = -\sqrt{\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\tanh\left[\frac{1}{2}\sqrt{-\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{15}$$

$$u_3(x,t) = \sqrt{\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\coth\left[\frac{1}{2}\sqrt{-\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{16}$$

$$u_4(x,t) = -\sqrt{\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\coth\left[\frac{1}{2}\sqrt{-\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{17}$$

where w is an arbitrary constant.

(ii) For $f(\zeta) = tg\ \zeta or\ ctg\ \zeta$

$$a + 2bA_0 = 0 \tag{18}$$

$$w - 6aA_0 - 6bA_1^2 - 6bA_0^2 - 8ck^2 = 0 \tag{19}$$

$$w - 6aA_0 - 6bA_0^2 - 2ck^2 = 0 \tag{20}$$

We have four unknown A_0, A, k, w and three equations and so may choose w as a free parameter. From (18)–(20), the other variables are then found to be

$$A_0 = -\frac{a}{2b} \tag{21}$$

$$A_1 = \pm\sqrt{\frac{-1}{2b}\left(w + \frac{3a^2}{2b}\right)} \tag{22}$$

$$k = \pm\sqrt{\frac{1}{2c}\left(w + \frac{3a^2}{2b}\right)} \tag{23}$$

Considering tg$(-\xi) = -$ tg ξ and ctg$(-\xi) = -$ctg ξ, the solitary wave solutions of (1) read

$$u_5(x,t) = \sqrt{-\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\, tg\left[\frac{1}{2}\sqrt{\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{24}$$

$$u_6(x,t) = -\sqrt{-\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\, tg\left[\frac{1}{2}\sqrt{\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{25}$$

$$u_7(x,t) = \sqrt{-\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\, ctg\left[\frac{1}{2}\sqrt{\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{26}$$

$$u_8(x,t) = -\sqrt{-\frac{1}{2b}\left(w + \frac{3a^2}{2b}\right)}\, ctg\left[\frac{1}{2}\sqrt{\frac{2}{c}\left(w + \frac{3a^2}{2b}\right)}(x - wt)\right] - \frac{a}{2b} \tag{27}$$

where w is an arbitrary constant.

Obviously, the solitary wave solutions of the mKdV equation can be obtained when $a = 0$.

3. THE DIRECT METHOD

In order to obtain the other type of solutions of (1), after integrating (3) once with respect to ξ, we have

$$-wu + 3au^2 + 2bu^2 + ck^2\, u_{\xi\xi} = \rho \tag{28}$$

where ρ is an integration constant. We assume a solution of (1) in the form

$$u(\xi) = \frac{B_1\, \mathrm{sech}^2\left(\frac{1}{2}\xi\right)}{4 + B_2\, \mathrm{sech}^2\left(\frac{1}{2}\xi\right)} + B_0 \tag{29}$$

where B_1 and B_2 are constants to be determined later.

Substituting equation (28) into equation (29), by equating the cofficients of corresponding hyperbolic sech functions to zero to make the result an identity, we arrive at the following four algebraic equations:

$$2bB_0^1 + 3aB_0^2 - wB_0 = \rho \tag{30}$$

$$ck^2 + 6aB_0 + 6bB_0^2 - w = 0 \tag{31}$$

$$(3a + 6bB_0)B_1 - 3ck^2(2 + B_2) = 0 \tag{32}$$

$$2bB^2 - 6ck^2 + (2 + B_2)(3a + 2bB_0)\, A - ck^2(2 + B_2)^2 - 2ck^2 = 0 \tag{33}$$

If B_0 is a real solution of (30), then there exist two cases to solve (31)–(33).
Case 1:

$$\Delta_1 = 3a + 2bB_0 \neq 0, \qquad \Delta_2 = -w + 6bB_0^2 + 6aB_0 < 0 \tag{34}$$
$$\Delta_3 = a^2 + 6w - 2b^2B_0^2 - 2abB_0 > 0$$

In this case, the solutions of (31)–(33) are given by

$$k = \pm\sqrt{-\frac{\Delta_2}{c}} \tag{35}$$

$$B_1 = \pm\frac{-6\sqrt{2}\Delta_2|\Delta_1|}{\sqrt{\Delta_3}\,\Delta_1} \tag{36}$$

$$B_2 = -2 \pm \frac{2\sqrt{2}|\Delta_1|}{\sqrt{\Delta_3}} \tag{37}$$

Case 2:

$$\Delta_1 = 0, \qquad b > 0, \qquad \Delta_4 = -w + 3aB_0 < 0 \tag{38}$$

In this case, the solutions of (31)–(33) are given by

$$k = \pm\sqrt{-\Delta_4} \tag{39}$$

$$B_1 = \pm\sqrt{-\frac{4}{b}\Delta_4} \tag{40}$$

$$B_2 = -2 \tag{41}$$

Thus the solitary wave solutions of (1) read as follows. *Case 1*:

$$u_9(x, t) = \frac{-(6\sqrt{2}\Delta_2|\Delta_1|/\sqrt{\Delta_2}\Delta_1)\,\text{sech}^2(1/2\sqrt{-(\Delta_2/c)}\xi)}{4 + (-2 + 2\sqrt{2}|\Delta_1|/\sqrt{\Delta_3})\,\text{sech}^2(1/2\sqrt{-(\Delta_2/c)}\xi)} + B_0 \tag{42}$$

$$u_{10}(x, t) = \frac{(6\sqrt{2}\Delta^2|\Delta_1|/\sqrt{\Delta_3}\Delta_1)\,\text{sech}^2(1/2\sqrt{-(\Delta_2/c)}\xi)}{4 + (-2 - 2\sqrt{2}|\Delta_1|/\sqrt{\Delta_3})\,\text{sech}^2(1/2\sqrt{-(\Delta_2/c)}\xi)} + B_0 \tag{43}$$

with $\Delta_1 \neq 0$, $\Delta_2 < 0$, $\Delta_3 > 0$.

Case 2:

$$u_{11}(x, t) = \sqrt{-\frac{2}{b}\Delta_4}\,\text{sech}(\sqrt{-\Delta_4}\zeta) + B_0 \tag{44}$$

$$u_{12}(x, t) = -\sqrt{-\frac{2}{b}\Delta_4}\,\text{sech}(\sqrt{-\Delta_4}\zeta) + B_0 \tag{45}$$

with $\Delta_1 = 0$, $\Delta_4 < 0$, $b > 0$.

Obviously, the solitary wave solutions of the KdV equation and the mKdV equation can be obtained respectively when $a = 0$ or $b = 0$.

REFERENCES

Coffey, . (1990).
Dey, . (1986).
Konno, K., and Ichikawa, Y. H. (1974). *Journal of the Physical Society of Japan*, **37**, 1631.
Lou, S. Y., and Chen, L. L. (1994). *Mathematical Methods in the Applied Sciences*, **17**, 339.
Mohamad, M. N. B. (1992). *Mathematical Methods in the Applied Sciences*, **15**, 73.
Miura, R. M. (1968). *Journal of Mathematical Physics*, **9**, 1202.
Narayanamurti, V., and Varma, C. M. (1970). *Physical Review Letters*, **25**, 1105.
Tappert, F. D., and Varma, W. (1970). *Physical Review Letters*, **25**, 1108.
Wadati, M. (1975a). *Journal of the Physical Society of Japan*, **38**, 673.
Wadati, M. (1975b). *Journal of the Physical Society of Japan*, **38**, 681.